Recitation 4. October 1

Focus: bases, and the four fundamental subspaces.

A ***basis*** of a vector space V is a set of vectors $v_1, \dots, v_n$ that I) span and II) are linearly independent. The number of vectors in a basis (which is always the same) is the ***dimension*** of the vector space.

Let A be an $m \times n$ matrix. The ***four fundamental subspaces*** of A are: I) the ***nullspace*** $N(A) \subset \mathbb{R}^n$, II) the ***column space*** $C(A) \subset \mathbb{R}^m$, III) the ***row space*** $C(A^T) \subset \mathbb{R}^n$, and IV) the ***left nullspace*** $N(A^T) \subset \mathbb{R}^m$.

1. Determine if each of the following is a basis for the given vector space. If it is not, add or remove vectors to make it one:

   (a) The set of vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ in $\mathbb{R}^3$.

   (b) The set of vectors $\begin{bmatrix}2\\4\\0\\3\end{bmatrix}, \begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}6\\0\\0\\0\end{bmatrix}$ in $\mathbb{R}^4$

   (c) The set of vectors $\begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\6\end{bmatrix}, \begin{bmatrix}4\\2\\5\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$ in $\mathbb{R}^3$.

Solution: (a) These vectors do form a basis. The reduced row echelon form is the identity matrix. In particular, each row has a pivot, so these vectors span, and there are no columns without pivots so there are no linear dependencies.

(b) No these vectors do not form a basis: there are 3 and we know $\mathbb{R}^4$ has dimension 4. To complete the set to a basis, we may calculate as follows. Up to swapping columns, we can write these vectors as the column space of the matrix (in reduced row echelon form)

$$\begin{bmatrix}6 & 1 & 2\\0 & -1 & 4\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$$

from which we see that there are no linear dependencies and there are pivots in rows 1, 3, and 4. Thus adding a column with a pivot in row 2 will make the column space all of $\mathbb{R}^4$. One choice is $v = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$.

(c) No these vectors do not form a basis: there are 4 of them and $\mathbb{R}^3$ has dimension 3. There must therefore be at least one linear dependence. To find it we use row elimination on the matrix whose columns are the above vectors. The row reduction is

$$\begin{bmatrix}1 & -3 & 4 & 2\\-3 & 1 & 2 & 1\\2 & 6 & 5 & 2\end{bmatrix} \xrightarrow{r_2+3r_1, r_3-2r_2} \begin{bmatrix}1 & -3 & 4 & 2\\0 & -8 & 14 & 7\\0 & 12 & -3 & -2\end{bmatrix} \xrightarrow{r_3+\frac{3}{2}r_2} \begin{bmatrix}1 & -3 & 4 & 2\\0 & -8 & 14 & 7\\0 & 0 & 18 & \frac{17}{2}\end{bmatrix}$$

from which we see the last column is linearly dependent on the first three.

2. Recall that for two vector subspaces V, W in $\mathbb{R}^n$, their sum is $V + W = \{v + w \mid v \in V \text{ and } w \in W\}$, and their intersection is $V \cap W = \{v \mid v \text{ is in both } V, W\}$. Let

$$V = \text{Span}\left\{\begin{bmatrix}-1\\0\\1\\4\end{bmatrix}, \begin{bmatrix}2\\6\\0\\-3\end{bmatrix}, \begin{bmatrix}1\\12\\2\\0\end{bmatrix}\right\} \qquad W = \text{Span}\left\{\begin{bmatrix}0\\-3\\1\\-3\end{bmatrix}, \begin{bmatrix}-1\\6\\3\\9\end{bmatrix}\right\}.$$

Find a basis for $V + W$ and $V \cap W$.

Solution: By definition, the set of all 5 vectors is a spanning set for $V+W$. We can proceed by row reduction on the matrix whose columns are the above five vectors in order to find any linear dependencies. The row reduction proceeds as:

$$\begin{bmatrix} -1 & 2 & 1 & 0 & -1 \\ 0 & 6 & 12 & -3 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 4 & -3 & 0 & -3 & 9 \end{bmatrix} \xrightarrow{r_3+r_1, r_4+4r_1} \begin{bmatrix} -1 & 2 & 1 & 0 & -1 \\ 0 & 6 & 12 & -3 & 6 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 5 & 4 & -3 & 5 \end{bmatrix} \xrightarrow{-r_1, \frac{1}{6}r_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 01 \\ 0 & 1 & 2 & -\frac{1}{2} & 1 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 5 & 4 & -3 & 5 \end{bmatrix}$$

$$\xrightarrow{r_3-2r_2, r_4-5r_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -6 & -\frac{1}{2} & 0 \end{bmatrix} \xrightarrow{-r_3, r_4+6r_3} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -\frac{25}{2} & 0 \end{bmatrix}$$

from which we see there is exactly one linear dependence: the fifth vector is a linear combination of the first two. From this we conclude the first four vectors form a basis of $V + W$, and the fifth vector is a basis for $V \cap W$ (since it is the only column that lies in both).

3. Use Gauss-Jordan elimination find a basis of each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & -3 & -1 & 2 \\ 2 & -5 & 2 & 5 \\ -3 & 9 & 4 & -3 \end{bmatrix}.$$

What is the dimension of each?

Solution: The reduced row echelon form is

$$\begin{bmatrix} 1 & -3 & -1 & 2 \\ 2 & -5 & 2 & 5 \\ -3 & 9 & 4 & -3 \end{bmatrix} \xrightarrow{r_2-2r_1, r_3+3r_1} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -28 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

From this we can see

I) There are three pivots in the first three columns, so the column space has dimension three and a basis for it is

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

II) The nullspace is one dimensional (the above shows the rank is $r = 3$ and $\dim N(A) + r = 4$). It is spanned by

$$\begin{bmatrix} 28 \\ 11 \\ -3 \\ 1 \end{bmatrix}$$

III) There are no linear dependencies among the rows, therefore the row space has dimension 3 and is spanned by

$$\begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 4 \\ -3 \end{bmatrix}$$

IV) The left nullspace has dimension $m - r = 3 - 3 = 0$. It has basis the empty set of vectors.

4. Let B be a square matrix such that $B^T = B^{-1}$. Show that the columns of B are (pairwise) orthogonal and have length 1. A matrix with this property is said to be ***orthogonal***.

Solution: If we label the columns of B by $\mathbf{c_1}, \ldots \mathbf{c_n}$ we can write

$$B = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \ldots & \mathbf{c_n} \end{bmatrix} \qquad\qquad B^T = \begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \vdots \\ \mathbf{c_n} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I = B^{-1}B = B^T B = \begin{bmatrix} \mathbf{c_1 \cdot c_1} & \mathbf{c_1 \cdot c_2} & \ldots & \mathbf{c_1 \cdot c_n} \\ \mathbf{c_2 \cdot c_1} & \mathbf{c_2 \cdot c_2} & \ldots & \mathbf{c_2 \cdot c_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c_n \cdot c_1} & \mathbf{c_n \cdot c_2} & \ldots & \mathbf{c_n \cdot c_n} \end{bmatrix}.$$

From which we see

$$\mathbf{c_i \cdot c_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

which shows that the columns are pairwise orthogonal, and each has length $\|\mathbf{c_i}\| = \sqrt{\mathbf{c_i \cdot c_i}} = \mathbf{1}$.